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The entanglement dynamics of the bipartite quantum system: toward entanglement sudden death

Wei Cui^{1,2}, Zairong Xi¹ and Yu Pan^{1,2}

¹ Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China ² Graduate University of Chinese Academy of Sciences, Beijing 100039, People's Republic of China

E-mail: zrxi@iss.ac.cn

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Abstract

We investigate the entanglement dynamics of the bipartite quantum system between two qubits with the dissipative environment. We begin with the standard Markovian master equation in the Lindblad form and the initial state which is prepared in the extended Werner-like state: $\rho_{AB}^{\Phi}(0)$. We examine the conditions for entanglement sudden death (ESD) and calculate the corresponding ESD time by the Wootters concurrence. We observe that ESD is determined by the parameters such as the mean occupation number of the environment *N*, amount of initial entanglement α and the purity *r*. For N = 0, we get the analytical expressions of both ESD condition and ESD time. For N > 0 we give a theoretical analysis that ESD always occurs, and simulate the concurrence as a function of $\gamma_0 t$ and one of the parameters N, α and r.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Entanglement is responsible for the most counterintuitive aspects of quantum mechanics [1], and motivated many philosophical discussions in the early days of quantum physics [2]. Recently it has been regarded as a resource for quantum information processing [3]. In fact, entanglement is one of the key ingredients for quantum teleportation [4–6], quantum cryptography [7] and is believed to be the origin of the power of quantum computers, etc. However, a quantum system used in quantum information processing inevitably interacts with the surrounding environments (or the thermal reservoirs), which take the pure state of the quantum system into a mixed state [8]. Thus, it is an important subject analyzing the entanglement decay induced by the unavoidable interaction of the interested systems with

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the environment [9]. In the one-party quantum system, this process is called decoherence and various methods have been proposed for reducing this unexpected effect [10-13]. In multiparty systems with non-local quantum correlations much interest has been arisen in the dynamics of entanglement. For example, entanglement sudden death (ESD), which means that it disappears at a finite time, was discovered by Yu and Eberly [14, 15]. It differs remarkably from the single qubit coherent evolution. This interesting phenomenon has been experimentally observed for entanglement photon pairs [16] and atomic ensembles [17].

ESD puts a limitation on the time when entanglement must be exploited. The evolution of the entanglement and ESD have been analyzed and various interesting results obtained [18–28]. Ikram *et al* [23] investigated the time evolution of entanglement of various entangled states of a two-qubit system. For four different initial states they analyzed the entanglement sudden death conditions and time. From the simulation they got the conclusion that the ESD always exists except for the vacuum reservoir. However, it is still a question when the ESD occurs and what ESD time is in the general cases and universal initial states. The aim of this paper is to discuss the above problems for the standard Markovian master equation in the Lindblad form and the initial states, the extended Werner-like states: $\rho_{AB}^{\Phi}(0)$.

This paper is organized as follows. We first introduce the Wootters' concurrence [29] and the extended Werner-like states: $\rho_{AB}^{\Phi}(0)$. In section 3 we give a standard Markovian master equation [30, 31]. The master equation is equivalent to a first-order coupled differential equations when the initial state is the extended Werner-like state $\rho_{AB}^{\Phi}(0)$. In sections 4, we analyze the ESD conditions and ESD time for the initial state $\rho_{AB}^{\Phi}(0)$. Conclusions and prospective views are given in section 5.

2. Concurrence and initial states

A useful measure of entanglement is the Wootters' concurrence [29]. For a bipartite system described by the density matrix ρ , the concurrence $C(\rho)$ is

$$\mathcal{C}(\rho) = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}), \tag{1}$$

where $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are the eigenvalues (with λ_1 the largest one) of the 'spin-flipped' density operator ζ , and

$$\zeta = \rho \left(\sigma_{y}^{A} \otimes \sigma_{y}^{B} \right) \rho^{*} \left(\sigma_{y}^{A} \otimes \sigma_{y}^{B} \right), \tag{2}$$

where ρ^* denotes the complex conjugate of ρ and σ_y is the usual Pauli matrix. C ranges in magnitude from 0 for a disentanglement state to 1 for a maximally entanglement state.

The general structure of an 'X' density matrix [14, 15] is as follows:

$$\hat{\rho} = \begin{pmatrix} x & 0 & 0 & v \\ 0 & y & u & 0 \\ 0 & u^* & z & 0 \\ v^* & 0 & 0 & w \end{pmatrix},$$
(3)

with x, y, z, w real positive and u, v complex quantities. Such states are general enough to include states such as the Werner states, the Bell states, *et al.*. A remarkable aspect of the 'X' states is that the initial 'X' structure is maintained during the Lindblad master equation evolution. This particular form of the density matrix allows us to analytically express the concurrence at time *t* as [14]

$$C_{\rho}^{X}(t) = 2\max\{0, |u| - \sqrt{xw}, |v| - \sqrt{yz}\}.$$
(4)

In the present work, we will analyze in detail the two-qubit entanglement dynamics in Markovian environment starting from the initial 'X' states defined in equation (3). We will



Figure 1. The initial entangled area (colored) of the extended Werner-like states for $\alpha \in [0, 2\pi]$. The blue line is $r = \frac{1}{1+2|\sin(2\alpha)|}$.

examine the exact ESD time and entanglement evolution for two kinds of special states, the extended Werner-like states,

$$\rho_{AB}^{\Phi}(0) = r |\Phi\rangle_{ABAB} \langle\Phi| + \frac{1-r}{4} \mathbf{I}_{AB},$$
(5)

where r is the purity of the initial states, I_{AB} is the 4 × 4 identity matrix and

$$\Phi_{AB} = (\cos(\alpha)|10\rangle + \sin(\alpha)|01\rangle)_{AB},\tag{6}$$

with α measuring the amount of initial entanglement. The state $\rho_{AB}^{\Phi}(0)$ has the following form:

$$\rho_{AB}^{\Phi}(0) = \begin{pmatrix} \frac{1-r}{4} & 0 & 0 & 0\\ 0 & r\cos^2(\alpha) + \frac{1-r}{4} & r\sin(\alpha)\cos(\alpha) & 0\\ 0 & r\sin(\alpha)\cos(\alpha) & r\sin^2(\alpha) + \frac{1-r}{4} & 0\\ 0 & 0 & 0 & \frac{1-r}{4} \end{pmatrix}.$$
 (7)

Obviously, the state in equation (5) is reduced to the standard Werner state when $\alpha = \pi/4$, and the Werner-like state becomes a totally mixed state for r = 0, and the well-known Bell state for r = 1. Note that the extended Werner-like state $\rho_{AB}^{\Phi}(0)$ contains both the separate state and the entangled state. According to the Peres criterion [32], when $\frac{1}{1+2|\sin 2\alpha|} \leq r \leq 1$ the extended Werner-like state would be entangled, otherwise it would be separated. Figure 1 shows the entangled and separated areas.

3. The master equation

The standard two-qubit Markovian master equation is of the following Lindblad form [8, 30, 31]:

$$\frac{d\rho}{dt} = \frac{\gamma_0(N+1)}{2} \sum_{j=1}^{2} \left\{ 2\sigma_j^- \rho \sigma_j^+ - \sigma_j^+ \sigma_j^- \rho - \rho \sigma_j^+ \sigma_j^- \right\} + \frac{\gamma_0 N}{2} \sum_{j=1}^{2} \left\{ 2\sigma_j^+ \rho \sigma_j^- - \sigma_j^- \sigma_j^+ \rho - \rho \sigma_j^- \sigma_j^+ \right\},$$
(8)

3

coupled differential equations:

where $N = 1/(e^{\frac{\hbar\omega_0}{K_BT}} - 1)$, the mean occupation number of the environment oscillators, γ_0 is the spontaneous emission rate. As introduced before, we assume the initial state to be an 'X' state. When substituting (3) into the master equation (8) we obtain the following first-order

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \\ \dot{w}(t) \end{pmatrix} = \begin{pmatrix} -2\gamma_0(N+1) & \gamma_0 N & \gamma_0 N & 0 \\ \gamma_0(N+1) & -\gamma_0(2N+1) & 0 & \gamma_0 N \\ \gamma_0(N+1) & 0 & -\gamma_0(2N+1) & \gamma_0 N \\ 0 & \gamma_0(N+1) & \gamma_0(N+1) & -2\gamma_0 N \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix},$$
(9) and

and

$$\dot{u}(t) = -(1+2N)\gamma_0 u(t), \qquad \dot{v}(t) = -(1+2N)\gamma_0 v(t).$$
(10)

The solution of the previous master equation can be found by solving the system of differential equations. The reduced density matrix elements x(t), y(t), z(t), w(t), u(t) and v(t) are given

(i)
$$N > 0$$

$$\begin{aligned} x(t) &= c_1 \frac{N}{N+1} - c_2 \Gamma^2(t) - c_4 \gamma_0 N \Gamma(t), \\ y(t) &= c_1 + c_2 \Gamma^2(t) + c_3 \Gamma(t), \\ z(t) &= c_1 + c_2 \Gamma^2(t) - c_3 \Gamma(t) - c_4 \gamma_0 \Gamma(t), \\ w(t) &= c_1 \frac{N+1}{N} - c_2 \Gamma^2(t) + c_4 \gamma_0 (N+1) \Gamma(t), \\ u(t) &= c_5 \Gamma(t), \\ v(t) &= c_6 \Gamma(t) \end{aligned}$$
(11)

(ii) N = 0

$$\begin{aligned} x(t) &= d_4 \Upsilon^2(t), \\ y(t) &= d_2 \Upsilon(t) + d_3 \Upsilon(t) - d_4 \Upsilon^2(t), \\ z(t) &= -d_3 \Upsilon(t) - d_4 \Upsilon^2(t), \\ w(t) &= d_1 - d_2 \Upsilon(t) + d_4 \Upsilon^2(t), \\ u(t) &= d_5 \Upsilon(t), \\ v(t) &= d_6 \Upsilon(t) \end{aligned}$$
(12)

where $\Gamma(t) = e^{-(1+2N)\gamma_0 t}$, $\Upsilon(t) = e^{-\gamma_0 t}$, and the coefficients $c_1, c_2, c_3, c_4, c_5, c_6$ in equation (11), and $d_1, d_2, d_3, d_4, d_5, d_6$ in equation (12) are determined by the corresponding initial conditions.

4. Entanglement dynamics with the initial conditions

4.1. N > 0

We now analyze the entanglement dynamics. Starting from the initial states $\rho_{AB}^{\Phi}(0)$ in equation (7), the coefficients of equation (11) are determined as

$$c_1 = \frac{N(N+1)}{(2N+1)^2},$$

$$c_2 = \frac{N(N+1)}{(2N+1)^2} - \frac{1-r}{4}$$

4



Figure 2. Concurrence of $C^{\Phi}_{\rho}(t)$ versus 'N' and $\gamma_0 t$ ($r = 1, \alpha = \pi/4$).

$$c_{3} = \frac{1}{2(2N+1)^{2}} + \frac{r}{2}(\cos^{2}(\alpha) - \sin^{2}(\alpha)),$$

$$c_{4} = -\frac{1}{\gamma_{0}(2N+1)^{2}},$$

$$c_{5} = r\sin(\alpha)\cos(\alpha),$$

$$c_{6} = 0.$$
(13)

Then the concurrence of $\rho^{\Phi}_{AB}(t)$ is

$$C(\rho_{AB}^{\Phi}(t)) = 2\max\{0, |u(t)| - \sqrt{x(t)w(t)}\}.$$
(14)

So the ESD appears when

$$|u(t)| - \sqrt{x(t)w(t)} \leq 0 \Leftrightarrow u^2(t) - x(t)w(t) \leq 0.$$
⁽¹⁵⁾

Obviously,

$$t \to +\infty,$$
 $u^2(t) - x(t)w(t) = -\frac{N^2(N+1)^2}{(2N+1)^4} < 0,$

and

$$t = 0,$$
 $u^{2}(t) - x(t)w(t) = r^{2}\sin^{2}(\alpha)\cos^{2}(\alpha) - \frac{(1-r)^{2}}{16}.$ (16)

Thus, ESD always occurs if the initial state is entangled, which means $\frac{1}{1+2|\sin 2\alpha|} \leq r \leq 1$, according to the Peres criterion.

Note that Ikram *et al* [23] found that when the mean thermal photon number is not zero, in the thermal reservoir the entanglement sudden death always happens from the simulation. However here we get the same result from the theoretic analysis. In figures 2, 3 and 4 we simulate the concurrence as a function of $\gamma_0 t$ and one of the parameters N, α and r, respectively. Figure 2 is the concurrence $C_{\rho}^{\Phi}(t)$ as a function of $\gamma_0 t$ and N, fixing the purity r = 1 and initial degree of entanglement $\alpha = \frac{\pi}{4}$, the Bell-like states. It shows that ESD time is affected by N. The smaller the N, the longer the ESD time. In figure 3, we plot the concurrence $C_{\rho}^{\Phi}(t)$ as a function of $\gamma_0 t$ and initial entanglement α when r = 1. When $\alpha = \frac{\pi}{4}$ or $\alpha = \frac{3\pi}{4}$ the initial



Figure 3. Concurrence of $C_{\rho}^{\Phi}(t)$ versus α and $\gamma_0 t$, (r = 1, N = 0.1).



Figure 4. Concurrence $C_{\rho}^{\Phi}(t)$ versus 'r' and $\gamma_0 t$, ($\alpha = \pi/4$, N = 0.25).

states are reduced to Bell-like states. The ESD time is sensibly affected by α . Figure 4 is the concurrence as a function of $\gamma_0 t$ and purity *r*. As exemplified before, when $0 \le r \le \frac{1}{1+2|\sin(2\alpha)|}$ the initial state is separated, so we choose *r* from 1/3 to 1. It shows that the larger the purity *r* the longer the ESD time.

Here we study the entanglement dynamics of the bipartite quantum system in the global environment effect. However, [26] had shown that under the classical noise effect entanglement may experience a sudden death process even if the local coherence of one participating particle is well preserved and the other one decays to zero asymptotically. How do the local thermal reservoirs influence the entanglement dynamics? To our knowledge, the master equation needs to be reconstructed and the local environment effect embodied by the spectral density of the thermal reservoir $J(\omega, T)$. The one-body decoherence dynamics was studied in [12, 33] under the local environment effect. Whether does the multipartite entanglement dynamics under the local thermal reservoirs hold under the classical noise effect proved by Yu *et al* [26]? We will study it in our further work.

4.2. N = 0

The coefficients in equation (12) have the form

 $d_1 = 1$,

6



Figure 5. The ESD area of the extended Werner-like states $\rho_{AB}^{\Phi}(t)$ with N = 0 for $\alpha \in [0, 2\pi]$. The red line is $r = \frac{-1+\sqrt{1+4\sin^2(2\alpha)}}{2\sin^2(2\alpha)}$, and the blue line is $r = \frac{1}{1+2|\sin(2\alpha)|}$.

Table 1. ESD condition and time for the initial state $\rho_{AB}^{\Phi}(0)$ at N = 0.

α	$(0, 2\pi)$
Condition	$\frac{1}{1+2 \sin(2\alpha) } < r < \frac{-1+\sqrt{1+4\sin^2(2\alpha)}}{2\sin^2(2\alpha)}$
Time	$\gamma_0 t^* = \ln(1-r) - \ln[2(1 - \sqrt{r + r^2 \sin^2(2\alpha)})]$

$$d_{2} = 1,$$

$$d_{3} = -\frac{1-r}{2} - r \sin^{2}(\alpha),$$

$$d_{4} = \frac{1-r}{4},$$

$$d_{5} = r \sin(\alpha) \cos(\alpha),$$

$$d_{6} = 0.$$
(17)

Thus

$$u^{2}(t) - x(t)w(t) \leq 0$$

$$\Rightarrow \Upsilon^{2}(t) - \frac{4}{1-r}\Upsilon(t) + \frac{4}{1-r} - \frac{16r^{2}}{(1-r)^{2}}\sin^{2}(\alpha)\cos(\alpha) \geq 0.$$
(18)

So the ESD occurs when

$$\frac{1}{1+2|\sin(2\alpha)|} < r < \frac{-1+\sqrt{1+4\sin^2(2\alpha)}}{2\sin^2(2\alpha)},\tag{19}$$

and the ESD time is

$$\gamma_0 t^* = \ln(1-r) - \ln\left[2(1-\sqrt{r+r^2\sin^2(2\alpha)})\right].$$
(20)

When $\gamma_0 t \in [\gamma_0 t^*, +\infty)$, the concurrence $C_{AB}^{\Phi} = 0$. In figure 5 we plot the ESD area. If the initial state is pure, i.e., r = 1, we find that $x(t) \equiv 0$. Then the concurrence is $C(\rho_{AB}^{\Phi}(t)) = 2|u(t)| = |\sin(2\alpha)|e^{-\gamma_0 t}$, which implies the nonexistence of ESD. After we submitted our paper, a closely related work appeared in [28]. Ikram *et al* extended their former result [23] to a more general state for the zero-temperature limit and provided some

discussions on the thermal case. Compared with ours, there are three main differences. First, although both papers studied the same system, a two 2-level atom system, different initial states had been chosen. These two initial states are independent, and the results are complementary. Second, for pure 2-qubit entangled states in a thermal environment, [28] says that 'we see in plots of figure 3 that entanglement sudden death always happens for non-zero average photon number in the two cavities and entanglement sudden death time depends upon the initial preparation of the entangled states'. Here, a theoretical analysis is provided by equations (14)–(16). Finally, in the vacuum reservoir we give the sufficient and necessary condition in equation (19) for the entanglement sudden death, and answer the question when the ESD occurs and what ESD time is. Furthermore, the ESD area relies on the initial purity r and initial entanglement α , see figure 5. Here we give a more detail examination and analysis.

The ESD condition and corresponding time for the initial state $\rho_{AB}^{\Phi}(0)$ at N = 0 are summarized in the following table.

5. Conclusions

In summary, we have analyzed the interesting phenomenon of entanglement sudden death determined by the dimensionless parameters $\hbar\omega_0/k_BT$, α and r. Sufficient conditions for ESD have been given for the initial state $\rho_{AB}^{\Phi}(0)$. We examine the conditions for ESD and calculate the corresponding ESD time by the Wootters' concurrence. We observe that ESD is determined by the parameters like N, amount of initial entanglement α and the purity r. For N = 0, we get the analytical expressions of the ESD condition and ESD time. For N > 0 we give a theoretical analysis that ESD always occurs, and simulates the concurrence as a function of $\gamma_0 t$ and one of the parameters N, α and r. In fact, these above results can also be obtained for other 'X' initial states, for example $\rho_{AB}^{\Psi}(0) = r |\Psi\rangle_{ABAB} \langle \Psi| + \frac{1-r}{4} \mathbf{I}_{AB}$, where $\Psi_{AB} = (\cos(\alpha)|00\rangle + \sin(\alpha)|11\rangle_{AB}$.

We analyze the dynamic behavior of entanglement in the open quantum system, but our ultimate aim is to control the entanglement such that it be a resource for practical realization. Despite the noticeable progresses of entanglement, many fundamental difficulties still remain. One of the problem is ESD due to the interactions between the system and environment; the other is that as the *N*-particle increased, the entanglement becomes arbitrarily small, and therefore useless as a resource [34]. Here we indicate that preparing some initial states can help prolong the ESD time. However, we think that it is a long way of how to design some effective control field to make the *N*-particle large enough and protect the entanglement that makes it practically useful.

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